Mathematics (Linear Algebra and optimization)

UPF Masters of Bioinformatics for Health

Eduardo Eyras

1. Let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3$. Consider the vectors

$$u_1 = e_1 + e_3,$$

$$u_2 = e_1 + a e_2,$$

$$u_3 = 4e_1 + e_3,$$

where $a \in \mathbb{R}$. Consider $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a linear map such that

$$f(e_1) = u_1,$$

$$f(e_2) = u_2,$$

$$f(e_3) = u_3.$$

Find for which values $a \in \mathbb{R}$ the vectors u_1, u_2, u_3 form a basis for $\mathbb{R}^3$.

Consider $v = (1, 1, 1) \in \mathbb{R}^3$ a vector in the canonical basis. Give the representation of v in the basis u_1, u_2, u_3. State clearly for which values of $a \in \mathbb{R}$ this representation makes sense and explain what happens in the other cases.

Give a matrix representation A_f for f in the canonical basis, using the canonical basis in both the origin and target spaces.

Give a matrix representation B_f for f considering the canonical basis in the origin space and the basis u_1, u_2, u_3 in the target space.

Is A_f diagonalizable? Specify for which basis in the origin and the target spaces the linear map f can be represented with a diagonal matrix.

2. The Fibonacci sequence is an infinite sequence of integer numbers of the form:

$$F=\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\}$$

Any element of the sequence F_n can be defined by the recurrence: $F_n = F_{n-1} + F_{n-2}$, with initialization $F_1=1, F_0=0$. This recurrence can be expressed in matrix form as follows:

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is usually called the Fibonacci matrix. The general form for any term in the series can thus be written as:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

Use singular value decomposition $A=USV^T$ to calculate A^n. Use this decomposition to obtain the general form of an element of the sequence in terms of the golden-ratio (phi) φ:

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$

Note: a matrix raised to an integer power n is the product of n matrices, e.g. $A^3 = A\,A\,A$.

3. Consider a linear map f: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ with matrix representation

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that any such linear map that preserves the norm of vectors and the orthogonality between vectors in $\mathbb{R}^2$ is an orthonormal matrix that can be written in the form:

$$A = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

Note: Use the standard scalar product in $\mathbb{R}^2$:

$$< u, v >= u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2, \text{ where } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

and recall that the linear map on a vector $u \in \mathbb{R}^2$ is written as follows:

$$Au = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$