Exercises – Projections and Subspaces

1. Consider two vectors $u = (u_i,\ldots,u_n), v = (v_i,\ldots,v_n)$ in the vector space $\mathbb{R}^n$. We define the scalar (or inner) product in $\mathbb{R}^n$ between these vectors as

$$< u,v >= \sum_{i=1}^{n} u_i \cdot v_i \,.$$

And the norm of a vector in a vector space is generally defined as:

$$\| u \| = \sqrt{< u,u >}$$

Show the following properties for $\mathbb{R}^n$:

a) $\| cv \| = | c | \cdot \| v \|$

b) $v/\| v \|$ is a unit vector

c) All the vectors of the canonical basis in $\mathbb{R}^n$ are unit vectors.

a) The distance between two vectors $d(u,v) = \| u - v \|$ is the Euclidean distance.

2. Consider the vectors $u = (1,2,0), v = (-1,4,1)$ in $\mathbb{R}^3$.

a) Calculate the distance $d(u,v)$

b) Calculate unit vectors from them

3. Consider the vectors $u = (1,1), v = (-2,2), w = (2,2)$ in $\mathbb{R}^2$. Using the scalar product, determine whether these vectors are orthogonal or parallel to each other.

4. Given two vectors in a vector space, $u,v \in V$,using the definition of norm and the properties of the scalar product, show that $\| u+v \|^2 = \| u \|^2 + \| v \|^2$ if u and v are orthogonal, i.e. $u \perp v$ (Pythagoras' theorem).

5. Given the definition of an orthogonal projection of a vector u onto a vector v:

$$proj_v(u) = \frac{< v,u >}{\| v \|^2} v \,,$$

show that $\left(u - proj_v(u)\right) \perp proj_v(u)$. (The orthogonal projection of a vector decomposes the vector into two orthogonal parts).

6. Given the vector $u = (a,b,c)$, $u \in \mathbb{R}^2$, calculate the orthogonal projections of u onto the vectors of the canonical basis.

7. Consider $\mathbb{R}^2$ with the basis $B = \{\beta_1, \beta_2\} = \left\{ \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$.

a) Confirm that B is a basis (i.e. Linear independent set and Span(B)= $\mathbb{R}^2$)

b) Verify that this is not an orthogonal basis (basis vectors orthogonal to each other).

c) Can you find an orthogonal basis from them using the definition of orthogonal projection? (Gram-Schmidt orthogonalization).

8. Consider the following basis for $\mathbb{R}^3$:

$$B = \{\beta_1, \beta_2, \beta_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}.$$

a) Verify that this is not an orthogonal basis.

b) Transform this basis into an orthogonal basis using the definition of orthogonal projection (Gram-Schmidt orthogonalization).